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ON THE COMPLEXITY OF D-DIMENSIONAL VORONOI DIAGRAMS.(U)

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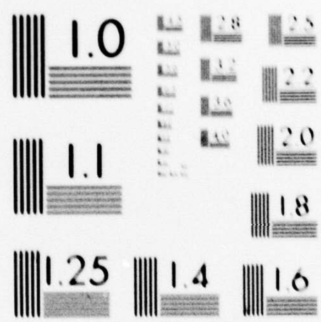

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ON THE COMPLEXITY OF d-DIMENSIONAL VORONOI DIAGRAMS

by

Victor Klee

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For n points $p_1, \ldots, p_n$ of Euclidean d-space E^d, the associated <u>Voronoi diagram</u> $V(p_1, \ldots, p_n)$ is a sequence $(P_1, \ldots, P_n)$ of convex polyhedra covering E^d, where p_i consists of all points of E^d that have p_i as a nearest point in the set $\{p_1, \ldots, p_n\}$. Voronoi diagrams in E^2 have been of interest because of their use by Shamos and others in providing efficient algorithms for a number of computational problems. The efficiency depends on the fact that the diagram itself can be computed efficiently (in time $O(n \log n)$ when $d = 2$). The present paper deals with the complexity of Voronoi diagrams based on n points of E^d. It is shown, in particular, that if $M_0(d,n)$ is the maximum number of vertices that such a diagram may have, then both the limit inferior and the limit superior of the sequence $(\lceil d/2 \rceil !\ M_0(d,n)/n^{\lceil d/2 \rceil})_{n=1,2,\ldots}$ are caught between 1 and 2 when d is even, and between $(\lceil d/2 \rceil e)^{-1}$ and 1 when d is odd.

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ON THE COMPLEXITY OF d-DIMENSIONAL VORONOI DIAGRAMS

VICTOR KLEE

Abstract Let $M_0(d,n)$ denote the maximum number of vertices of Voronoi diagrams based on n points of E^d. Then for each d, the cluster points of the sequence $(\lceil d/2\rceil!\, M_0(d,n)/n^{\lceil d/2\rceil})$ $n = 1,2$ all lie in the interval $[1,2]$ when d is even and in $[(\lceil d/2\rceil e^{-1},1]$ when d is odd.

Introduction For n points $p_1,\ldots, p_n$ of Euclidean d-space E^d, the associated Voronoi diagram $V(p_1,\ldots p_n)$ is a sequence $(P_1,\ldots P_n)$ of convex polyhedra covering E^d, where P_i consists of all points of E^d that have p_i as a nearest point in the set $\{p_1,\ldots p_n\}$. Thus

$$P_i = \{x \in E^d: \|x-p_i\| \leq \|x-p_j\| \text{ for all } j\} = \bigcap_{j\neq i} H_{ij},$$

where $$H_{ij} = \{x \in E^d: \langle p_j-p_i,\, x\rangle \leq \tfrac{1}{2}(\|p_j\|^2 - \|p_i\|^2)\}.$$

Note that H_{ij} is the closed halfspace which contains p_i and whose bounding hyperplane passes through the midpoint of the segment $[p_i, p_j]$ and is perpendicular to that segment.

For $0 \leq k < d$, let $\phi_k(p_1,\ldots,p_n)$ denote the number of sets S such that S is a k-dimensional face of at least one of the polyhedra P_i. Then $\phi_k(p_1,\ldots p_n)$ is a natural measure of the complexity of the diagram, and the cases $k = 0$ and $k = d-1$ are of special interest. Let $M_k(d,n)$ denote the

maximum of $\phi_k(p_1,\ldots p_n)$ as $(p_1,\ldots,p_n)$ ranges over all n-tuples of distinct points of E^d. A routine application of Euler's theorem shows

$$M_0(2,n) = 2n - 5 \text{ and } M_1(2,n) = 3n - 6 \quad \text{for all } n > 2.$$

Here it is proved that

(1) $$M_{d-1}(d,n) = \binom{n}{2} \quad \text{for } d \geq 3, \text{ all } n,$$

(2) $$1 \leq \liminf_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leq \limsup_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leq 2 \quad \text{for even } d = 2r,$$

(3) $$\frac{1}{re} < \liminf_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leq \limsup_{n\to\infty} \frac{M_0(d,n)}{n^r/r!} \leq 1 \quad \text{for odd } d = 2r - 1.$$

Our method can also be used to obtain inequalities for the other M_k's

__Theorems__ Not suprisingly, all our proofs are based on properties of neighborly polytopes. A d-polytope (that is, a bounded d-dimensional convex polyhedron) is said to be __neighborly__ if each set of $\lfloor d/2 \rfloor$ of its vertices is the vertex-set of a face. This implies that for $1 \leq j \leq \lfloor d/2 \rfloor$, each set of j vertices is the vertex set of a $(j-1)$- face. For discussions and constructions of neighborly polytopes, see Gale [4] and Grünbaum [5].

THEOREM 1 __If__ $1 \leq j \leq \lfloor d/2 \rfloor$ __then__ $M_{d-j+1}(d,n) = \binom{n}{j}$ __for all__ n.

Proof. The cases in which $n \leq d + 1$ are left to the reader. With $n \geq d + 2$, let $w_1,\ldots, w_{n-1}$ be the vertices of a neighborly d-polytope

Q in E^d such that the origin is interior to Q. Then for $1 \leq j \leq \lfloor d/2 \rfloor$, each j facets ($(d-1)$ - faces) of the polar polytope

$$Q^0 = \{x \in E^d : \langle w_i, x \rangle \leq 1 \text{ for } 1 \leq i < n\}$$

intersect in a $(d-j)$ - face of Q^0. For $1 \leq i < n$, let $p_i = 2w_i/\|w_i\|^2$. Then for each $x \in E^d$,

$$\langle w_i, x \rangle = 1 \iff \langle p_i, x \rangle = \tfrac{1}{2}\|p_i\|^2.$$

Let $p_n = 0$ and $(P_1, \ldots, P_n) = V(p_1 \ldots, p_n)$. Since the affine hulls of Q^0's facets are the sets of the form $\{x: \langle w_i, x \rangle = 1\}$ for $1 \leq i \leq n$, it follows that $P_n = Q^0$ and for $1 \leq i < n$ the intersection $P_i \cap P_n$ is a facet F_i of P_n.

Let $\underset{\sim}{I}$ $\langle$resp. $\underset{\sim}{I}'\rangle$ consist of all j-sets $I \subset \{1, \ldots, n\}$ such that $n \in I$ $\langle$resp $n \notin I\rangle$, and for each $I \in \underset{\sim}{I} \cup \underset{\sim}{I}'$ let $G_I = \bigcap_{i \in I} P_i$. If $I \in \underset{\sim}{I}$ then $G_I = \bigcap_{i \in I \sim \{n\}} F_i$, a $(d-j+1)$ - face of P_n. If $I \in \underset{\sim}{I}'$ then $G_I \cap P_n$ is a $(d-j)$ - face of P_n and for each $i \in I$ is the intersection with P_n of a $(d-j+1)$ - face of P_i. Since different members of $\underset{\sim}{I}$ $\langle$resp. $\underset{\sim}{I}'\rangle$ give rise to distinct sets G_I $\langle$resp. $G_I \cap P_n\rangle$, the stated conclusion follows.□

A polytope is <u>simplicial</u> if all its facets are simplices. It is known [4] that all neighborly d-polytopes are simplicial when d is even, and [5] that the number of facets of a simplicial neighborly d-polytope with n vertices is

$$\gamma(d,n) = \binom{n - \lfloor (d+1)/2 \rfloor}{n-d} + \binom{n - \lfloor (d+2)/2 \rfloor}{n-d}$$

McMullen [7] proved that $\gamma(d,n)$ is the maximum number of facets of d-polytopes with n vertices and hence, dually, of vertices of d-polytopes

with n facets.

A d-polyhedron is <u>simple</u> if it has at least one vertex and each of its vertices is incident to precisely d edges or, equivalently, to precisely d facets ((d-1)- faces). A d-dimensional Voronoi diagram $V(p_1,\ldots,p_n)$ is <u>simple</u> if it has at least one vertex and each vertex is incident to precisely $d + 1$ of the P_i's; this implies that all the P_i's are simple.

THEOREM 2 *If $p_1,\ldots p_n$ are distinct points of E^d such that the Voronoi diagram $V(p_1, \ldots p_n) = (P_1,\ldots P_n)$ is simple and u of the P_i's are unbounded, then $\phi_0(p_1,\ldots p_n) \le \gamma(d+1,n) + d - u$.*

Proof. The assertion is obvious when $d = 2$, so we assume $d > 2$. A theorem of Davis [2] then guarantees the existence of a real-valued convex function f on E^d such that each P_i is a set X which is maximal with respect to there being an affine function on E_d that agrees with f on X. The epigraph $\{(x,\tau): \tau \ge f(x)\}$ is a simple $(d+1)$ - polytope that has precisely n facets, u of which are unbounded. It then follows from an extension [6] of McMullen's theorem that the number of vertices of the epigraph, and hence of $V(p_1,\ldots, p_n)$, is at most $\gamma(d+1,n) + d - u$. □

THEOREM 3 *If $n > d + 1$ then $\gamma(d,n-1) \le M_0(d,n) < \gamma(d+1, n)$.*

Proof. To establish the lower bound, carry out the construction of Theorem 1 with a neighborly polytope Q that is simplicial. Then Q has $\gamma(d,n-1)$ facets, so $\gamma(d,n-1)$ is also the number of vertices of the polar polytope $Q^o = P_n$.

For the upper bound, note that whenever $p_1, \ldots, p_n$ are points of E^d (with $n > d$), they can be perturbed slightly so that the diagram $V(p_1,\ldots,p_n)$

becomes simple and its number of vertices does not decrease. (A formal proof can be based on a semicontinuity theorem of [3].) Then use Theorem 2, noting that the number of unbounded P_i's must exceed d. □

Note that

$$\gamma(d,n) = \frac{n}{n-r}\binom{n-r}{r} \qquad \text{for even } d = 2r$$

and

$$\gamma(d,n) = 2\binom{n-r}{r-1} \qquad \text{for odd } d = 2r-1.$$

Thus Theorem 3 yields the following corollary, which in turn implies (2).

COROLLARY 1 *For even* $d = 2r$ *and for* $n > d + 1$,

$$\frac{n-1}{n-1-r}\binom{n-1-r}{r} \le M_0(d,n) < 2\binom{n-1-r}{r}.$$

To establish (3) we use an idea of Preparata [8] in conjunction with some special neighborly polytopes.

THEOREM 4 *If* d *is odd*, $s > d$ *and* $t \ge 1$, *then* $M_0(d,s+t) \ge t\gamma(d-1,s-1)$.

Proof. Let $d = 2r+1$, and for each angle θ let

$$x(\theta) = (\sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta, \ldots, \sin r\theta, \cos r\theta) \in E^{2r}.$$

Let

$$C_r = \{x(\theta) : 0 \le \theta \le 2\pi\},$$

a simple closed curve on the sphere in E^{2r} that is centered at 0 and has radius $\sqrt{r}$. This curve was studied by Caratheodory [1] and also by Gale

[4], who observed that the convex hull con X is a neighborly $(2r)$- polytope for each finite set X of more than $2r$ points of C_r. Grünbaum [5] noted this is easily proved with the aid of Scott's identity [11] asserting that if $\delta(\theta_1,\ldots,\theta_d)$ is the determinant of the matrix whose i^{th} row consists of a 1 followed by $x(\theta_i)$, then

$$\delta(\theta_1,\ldots,\theta_d) = 2^{2r^2} \Pi_{1 \le i < j \le d} \sin \tfrac{1}{2}(\theta_j-\theta_i).$$

For $1 \le i \le d$, let $\alpha_i = 2\pi(i-1)/d$ and $w_i = x(\alpha_i)$. From neighborlines and a remark of Gale [4], and also from Scott's identity, it follows that the convex hull of the w_i's is a $(2r)$ - simplex. Since $\Sigma_1^d w_1 = 0$, the origin is interior to the simplex. For the given $s > d$, let $w_{d+1},\ldots, w_s$ be distinct points of $C_r \sim \{w_1,\ldots, w_d\}$. For $1 \le i < s$, let $p_i = 2w_i/\| w_i\|^2$, so that $\| p_i\| = 2/\sqrt{r}$, and let $p_s = 0$. With $(P_1,\ldots, P_s) = V(p_1,\ldots, p_s)$, P_s is the polar of the neighborly $(2r)$ - polytope con $\{w_1,\ldots, w_{s-1}\}$ and hence P_s has $\gamma(d-1,s-1)$ vertices. Let $q_i = (\sqrt{r}/2)\, p_i$ for $1 \le i \le s$, so that $q_s = 0$, $\| q_i\| = 1$ for $1 \le i < s$, and the polytope

$$K = \{x \in E^{2r}: \ \| x \| \le \|x - q_i\| \text{ for } 1 \le i < s\}$$

is equal to $(\sqrt{r}/2)P_s$.

Now let E^{2r} be embedded in E^d as a hyperplane through the origin, having a line Rz with $\| z \| = 1$ as orthogonal supplement. For $1 \le i \le t$ let $q_{s+i} = 2iz$. Let $(Q_1, \ldots, Q_{s+t})$ denote the Voronoi diagram $V(q_1,\ldots, q_{s+t})$. We prove $M_o(d, s+t) \ge t\gamma(d-1, s-1)$ by showing for $1 \le i \le t$ that Q_{s+i} has $\gamma(d-1, s-1)$ vertices in the hyperplane

$$J_i = E^{2r} + (2i-1)Z.$$

All points of J_i are equidistant from q_{s+i-1} and q_{s+i}, and are closer to these than to any other point of the set $\{q_s, \ldots, q_{s+t}\}$. The point $(2i-1)\,Z$ is closer to q_{s+i-1} and to q_{s+i} than to any other point of the set $\{q_1, \ldots, q_{s+t}\}$. Thus J_i contains a facet F of Q_{s+i}, and in fact

$$F = \bigcap_{1 \le k < s} (H_k \cap J_i)$$

where

$$H_k = \{x \in E^d : \| x - q_{s+i} \| \le \| x - q_k \|\}.$$

To see that F has the same number of vertices as K, note that there is a point $-\mu z$ such that F is the intersection with J_i of the convex cone formed by all rays that issue from $-\mu z$ and pass through points of K. The vertices of F are the intersections of J_i with the edges of the cone, and these in turn correspond to vertices of K. The existence of $-\mu z$, which can be deduced from the lemma below, depends on all the points $q_1, \ldots q_{s-1}$ having the same norm, and that was the reason for the special choice of neighborly polytopes in this construction. (Thus having $\| q_1 \| = \ldots = \| q_{s-1} \|$ appears to be essential here, though having these norms = 1 is merely a computational convenience.)

Lemma. Suppose E is a hyperplane through the origin in a Euclidean space, having a line Rz with $\| z \| = 1$ as orthogonal supplement. Suppose $q \in E$ with $\| q \| = 1$, and suppose $0 < \beta < \eta \le 2\beta$. Let

$$\psi = \frac{\eta(2\beta-\eta) + 1}{2} \quad \text{and} \quad \mu = \frac{\beta}{2\,\psi-1} = \frac{\beta}{\eta(2\beta-\eta)}.$$

Then for each point x of the hyperplane $E + \beta z$, the following two conditions are equivalent:

(i) $\| x - \eta z \| \le \| x - q \|$;

(ii) if x' is the point at which the segment $[-\mu z, x]$ intersects E, then $\| x' \| \le \| x' - q \|$.

To prove lemma, consider an arbitrary point $x \in E + \beta z$ — say $z = y + \beta z$ with $y \in E$. Consideration of similar triangles shows that $x' = \varepsilon y$ with $\varepsilon = \mu/(\mu+\beta) = 1/(2\psi)$. Using the facts that $\langle z, y \rangle = \langle z, q \rangle = 0$ and $\langle z, z \rangle = \langle q, q \rangle = 1$, both (i) and (ii) are seen to be equivalent to the inequality $\langle q, y \rangle \le \psi$. That settles the lemma and completes the proof of Theorem 4. □

CORRELLARY 2 <u>For odd</u> $d = 2r-1$ <u>and for</u> $n > d+1$,

$$\frac{n-r-1}{r+1} \frac{nr-r-1}{nr-r^2+1} \binom{\lceil nr/(r+1) \rceil - r}{r-1} < M_0(d,n) < \frac{n}{n-r} \binom{n-r}{r}.$$

Proof. Use Theorem 3 for the upper bound. For the lower bound, apply Theorem 4 with $s = \lceil nr/(r+1) \rceil$ and $t = \lfloor n/(r+1) \rfloor$, obtaining

$$M_0(d,n) \ge t\gamma(2r-2, s-1) = t \frac{s-1}{s-r} \binom{s-r}{r-1}$$

and hence the stated lower bound. From the latter it follows that

$$\liminf_{n \to \infty} M_0(d,n) \ge \frac{1}{r} \left(\frac{r}{r+1}\right)^r n^r > \frac{1}{re} n^r$$

thus settling (3).□

<u>Comments</u> For applications of Voronoi diagrams to problems of packing and covering in E^d, and for references to the earlier literature, see Rogers

[10]. In recent years, Voronoi diagrams in E^2 have been of interest because of their use by Shamos [12] and Shamos and Hoey [13] in providing efficient algorithms for a number of computational problems. For n points $p_1, \ldots, p_n$ of E^2, the diagram $(P_1, \ldots P_n)$ can be computed in time $O(n \log$ each P_i being output as its sequence of successive vertices. The same computation in E^d would in worst cases require time $\Omega(n^{\lceil d/2 \rceil})$ because of the possible number of vertices. However, it is unknown whether, in time bounded by some polynomial in d and n, one can compute the facets of the P_i's. For input $p_1, \ldots, p_n \in E^d$, the output would consist of n subsets $I_1, \ldots, I_n$ of $\{1, \ldots, n\}$ such that $i \in I_j$ if and only if the hyperplane $\{x: \| x - p_i \| = \| x - p_j \| \}$ contains a facet of P_j. By results of Reiss and Dobkin [9], this can be accomplished in polynomial time if and only if linear programming problems with d variables and n constraints can be solved in polynomial time.

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